

Wormhole geometries with conformal motions

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Exact solutions of traversable wormholes were recently found under the assumption of spherical symmetry and the existence of a *non-static* conformal symmetry. In this paper, we verify that in the case of the conformally symmetric spacetimes with a non-static vector field generating the symmetry, the conformal factor ψ can be physically interpreted in terms of a measurable quantity, namely, the tangential velocity of a massive test particle moving in a stable circular orbit in the spacetime. Physical properties of the rotational velocity of test particles and of the redshift of radiation emitted by ultra-relativistic particles rotating around these hypothetical general relativistic objects are further discussed. Finally, specific characteristics and properties of gravitational bremsstrahlung emitted by charged particles in geodesic motion in conformally symmetric wormhole geometries are also explored.

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I. INTRODUCTION

Wormholes are hypothetical shortcuts in spacetime [1, 2], and are primarily useful as “gedanken-experiments” and as a theoretician’s probe of the foundations of general relativity. These solutions are a specific example in solving the Einstein field equation in the reverse direction, namely, one first considers an interesting spacetime metric, then finds the matter source responsible for the respective geometry. In this manner, it was found that some of these solutions possess a peculiar property, namely “exotic matter,” involving a stress-energy tensor that violates the null energy condition, and a number of specific solutions have been found (see Ref. [3] and references therein, and Ref. [4] for a recent review). These geometries also allow closed timelike curves, with the respective causality violations [5].

A more systematic approach in searching for exact solutions, namely, by assuming spherical symmetry and the existence of a *non-static* conformal symmetry, was recently considered in [6]. Suppose that the vector ξ generates the conformal symmetry, then the metric \mathbf{g} is conformally mapped onto itself along ξ . This is translated by the following relationship

$$\mathcal{L}_\xi \mathbf{g} = \psi \mathbf{g}, \quad (1)$$

where \mathcal{L} is the Lie derivative operator and ψ is the conformal factor. Note that neither ξ nor ψ need to be static even though one considers a static metric. A wide variety of solutions with the exotic matter restricted to the throat neighborhood and with a cut-off of the stress-energy tensor at a junction interface were deduced, and particular asymptotically flat geometries were also found. The specific solutions were deduced by considering choices for the form function, an equation of state relating the energy density and the anisotropy, and phantom wormhole geometries were also explored.

The assumption of a static conformal symmetry, i.e., with a static vector ξ , considered in [7, 8] was found responsible for the singular solutions at the center. However, we emphasize that this is not problematic to wormhole physics, due to the absence of a center [6]. Indeed, the case of a non-static conformal symmetry ξ generating vector field ξ does not yield a singularity at $r = 0$ and a wide variety of solutions were found [8]. Therefore, since the origin is an allowed point, the analysis is in fact more general than wormholes. In this context, it is interesting to note that the physical properties and characteristics of conformal symmetries have also been applied to a wide variety of geometries in the literature [9, 10, 11], in particular, an exact analytical solution describing the interior of a charged strange quark star was found [9]; solutions were also explored in braneworlds [10]; and also in the context of the galactic rotation curves [11].

It is the purpose of this paper to consider the behavior of some observationally relevant physical quantities in the conformally symmetric wormhole geometry analyzed

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in detail in [6]. As a first step we consider the behavior of the massive test particles in stable circular orbits around the wormhole. Under the assumption of spherical symmetry we derive the basic equation describing the tangential velocity of the particle, which can be obtained as a function of the derivative with respect to the radial coordinate of the redshift function only. On the other hand, because the derivative of the redshift function can be expressed in terms of the conformal factor describing the basic *geometrical* properties of the wormhole, it follows that the conformal factor can be obtained directly from the tangential velocity. The conformal factor also determines the form function. The tangential velocity of particles in circular orbits around wormholes could be determined, at least in principle, from astronomical observations of the frequency shifts (both blue and red) of the light emitted by the charged particles orbiting the wormhole. Therefore this opens the possibility of a direct observational determination of the geometry of the conformally symmetric wormholes.

Another observationally important physical parameter is the total power radiated by charged particles, either spiraling towards the wormhole in circular orbits, or falling on radial paths. The process of electromagnetic radiation emission due to the gravitational acceleration of charged particles in geodesic motion is called gravitational bremsstrahlung. The measurement of this power from astronomical/astrophysical observations would allow the determination of the conformal factor, thus allowing a direct test of the wormhole geometry.

The present paper is organized as follows. We review the properties of the conformally symmetric wormholes in Section II. The relation between the tangential velocity of particles in stable circular orbits and the geometric properties of the spacetime is considered in Section III. We analyze the behavior of the rotational velocity of test particles and of the frequency shifts of light in Section IV. In Section V, we explore some physical properties and characteristics of gravitational bremsstrahlung emitted by charged particles in geodesic motion in conformally symmetric wormhole geometries. Finally, we discuss and conclude our results in Section VI.

II. CONFORMAL SYMMETRY AND WORMHOLE GEOMETRY

An alternative approach in searching for exact wormhole solutions [6] is based on the assumption that the spherically symmetric static spacetime possesses a conformal symmetry [7, 8]. It is interesting to note, as mentioned in the Introduction, that neither ξ nor ψ need to be static even though one considers a static metric, and therefore Eq. (1) takes the following form

$$g_{\mu\nu,\alpha} \xi^\alpha + g_{\alpha\nu} \xi^\alpha_{,\mu} + g_{\mu\alpha} \xi^\alpha_{,\nu} = \psi g_{\mu\nu}. \quad (2)$$

As in Ref. [6], we follow closely the assumptions made in Ref. [8], where the condition

$$\xi = \alpha(t, r) \partial_t + \beta(t, r) \partial_r, \quad (3)$$

is considered, and the conformal factor is static, i.e., $\psi = \psi(r)$.

The spacetime metric that will be considered, representing a spherically symmetric and static wormhole, is given by

$$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2 d\Omega^2, \quad (4)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$, and $\Phi(r)$ and $b(r)$ are arbitrary functions of the radial coordinate, r , denoted as the redshift function, and the form function, respectively [1]. The radial coordinate r possesses a minimum value at r_0 , representing the location of the throat of the wormhole, where $b(r_0) = r_0$, and consequently has the following range $r \in [r_0, +\infty)$. The proper circumference of a circle of fixed r is given by $2\pi r$. The redshift function $\Phi(r)$ is finite throughout the range of interest in order to avoid the absence of event horizons. The form function is also constrained by the flaring-out condition, $(b'r - b)/b^2 < 0$, which reduces to $b'(r_0) < 1$ at the throat.

Taking into account metric (4), then Eq. (2), without a loss of generality (see Ref. [6] for details), provides the following solutions

$$\xi = \frac{1}{2} k t \partial_t + \frac{1}{2} \psi(r) r \partial_r$$

and

$$b(r) = r[1 - \psi^2(r)], \quad (5)$$

$$\Phi(r) = \frac{1}{2} \ln(C^2 r^2) - k \int \frac{dr'}{r' \psi(r')}, \quad (6)$$

respectively, where k and C are constants. An interesting feature of these solutions that immediately stands out, by taking into account Eq. (5), is that the conformal factor is zero at the throat, i.e., $\psi(r_0) = 0$.

Note that the solutions given by Eqs. (5) and (6) impose the following condition, relating the form and redshift functions

$$\Phi'(r) = \frac{1}{r} \left(1 - \frac{k}{\sqrt{1 - b(r)/r}} \right). \quad (7)$$

Note that this relationship places a strong constraint on the specific choices of the wormhole geometries. A wide range of specific solutions were deduced in Ref. [6], by considering choices for the form function. Note that, in principle, one may also impose interesting choices for the redshift function, and consequently deduce the form function and the conformal factor.

It is rather important to emphasize that specific examples of asymptotically flat spacetimes were found in [6], by considering the case of $k = 1$ and normalizing the

integration constant in Eq. (6) by imposing the value $C^2 = 2$. Thus, in the analysis that follows, one may consider asymptotically flat spacetimes simply by imposing the above values for the respective constants.

The existence of conformal motions imposes strong constraints on the wormhole geometry, so that the stress-energy tensor components are written solely in terms of the conformal function [6], and take the following form

$$\rho(r) = \frac{1}{\kappa^2 r^2} (1 - \psi^2 - 2r\psi\psi'), \quad (8)$$

$$p_r(r) = \frac{1}{\kappa^2 r^2} (3\psi^2 - 2k\psi - 1), \quad (9)$$

$$p_t(r) = \frac{1}{\kappa^2 r^2} (\psi^2 - 2k\psi + k^2 + 2r\psi\psi'), \quad (10)$$

where $\kappa^2 = 8\pi$; $\rho(r)$ is the energy density, $p_r(r)$ is the radial pressure, and $p_t(r)$ is the lateral pressure measured in the orthogonal direction to the radial direction. Note that the conservation of the stress-energy tensor provides the following relationship

$$p_r' = \frac{2}{r} (p_t - p_r) - (\rho + p_r) \Phi'. \quad (11)$$

The NEC violation, for this case, is given by

$$\rho(r) + p_r(r) = \frac{1}{\kappa^2 r^2} [2\psi(\psi - k) - r(\psi^2)'], \quad (12)$$

which evaluated at the throat imposes the following condition $(\psi^2)' > 0$.

III. STABLE CIRCULAR ORBITS IN STATIC AND SPHERICALLY SYMMETRIC SPACETIMES

In the case of the conformally symmetric spacetimes with a non-static vector field generating the symmetry, the conformal factor ψ can be physically interpreted in terms of a measurable quantity, the tangential velocity of a massive test particle moving in a stable circular orbit in the spacetime described by the line element given by Eq. (4).

To verify this, consider the Lagrangian \mathcal{L} for a massive test particle, which reads

$$\mathcal{L} = \frac{1}{2} \left[-e^{2\Phi} \dot{t}^2 + \frac{\dot{r}^2}{1 - b/r} + r^2 \dot{\Omega}^2 \right], \quad (13)$$

where the overdot denotes differentiation with respect to the affine parameter s . Since the metric tensor coefficients do not explicitly depend on t and Ω , the Lagrangian (13) yields the following conserved quantities (generalized momenta):

$$-e^{2\Phi(r)} \dot{t} = E, \quad r^2 \dot{\Omega} = L, \quad (14)$$

where E is related to the total energy of the particle and L to the total angular momentum. With the use

of the conserved quantities, we obtain from Eq. (13) the geodesic equation for massive particles in the form

$$e^{2\Phi} \left(1 - \frac{b}{r} \right)^{-1} \dot{r}^2 + e^{2\Phi} \left(1 + \frac{L^2}{r^2} \right) = E^2. \quad (15)$$

This equation shows that the radial motion of the particles on a geodesic is the same as that of a particle with position dependent mass and with energy $E^2/2$ in ordinary Newtonian mechanics moving in the effective potential

$$V_{eff}(r) = e^{2\Phi} \left(\frac{L^2}{r^2} + 1 \right). \quad (16)$$

For the case of the motion of particles in circular and stable orbits the effective potential must satisfy the following conditions: a) $\dot{r} = 0$, representing circular motion; b) $\partial V_{eff}/\partial r = 0$, providing extreme motion; c) $\partial^2 V_{eff}/\partial r^2|_{extr} > 0$, translating a stable orbit. Conditions a) and b) immediately provide the conserved quantities as

$$E^2 = e^{2\Phi} \left(1 + \frac{L^2}{r^2} \right), \quad (17)$$

and

$$\frac{L^2}{r^2} = r\Phi' e^{-2\Phi} E^2, \quad (18)$$

respectively. Equivalently, these two equations can be rewritten as

$$E^2 = \frac{e^{2\Phi}}{1 - r\Phi'}, \quad L^2 = \frac{r^3 \Phi'}{1 - r\Phi'}. \quad (19)$$

The line element, given by Eq. (4), can be rewritten in terms of the spatial components of the velocity [12], where

$$v^2 = e^{-2\Phi} \left[\frac{1}{1 - b(r)/r} \left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\Omega}{dt} \right)^2 \right]. \quad (20)$$

For a stable circular orbit $\dot{r} = 0$, the tangential velocity of the test particle can be expressed as

$$v_{tg}^2 = r^2 e^{-2\Phi} \left(\frac{d\Omega}{dt} \right)^2 = e^{-2\Phi} r^2 \dot{\Omega}^2 / \dot{t}^2 = e^{2\Phi} \frac{L^2}{r^2 E^2}. \quad (21)$$

By using the constants of motion, we obtain the expression of the tangential velocity of a test particle in a stable circular orbit, given by

$$v_{tg}^2 = r\Phi'. \quad (22)$$

Thus, the rotational velocity of the test body is determined by the redshift function only. This expression which relates one of the metric components to the tangential velocity is an exact general relativistic expression valid for static and spherically symmetric spacetimes.

Note that for generic redshift functions, where $\Phi' < 0$, there are no stable circular orbits. The expression also imposes that $0 \leq r\Phi' < 1$. The specific case of a constant redshift function imposes that $v_{\text{tg}}^2 = 0$ for all values of r .

As a second consistency condition we require that the timelike circular geodesics in the wormhole geometry be stable. Let r_{eq} be a circular orbit and consider a perturbation of it of the form $r = r_{eq} + \delta$, where $\delta \ll r_{eq}$ [13]. Taking expansions of $V_{eff}(r)$ and $1/(1 - b(r)/r)$ about $r = r_{eq}$, it follows from Eq. (15) that

$$\ddot{\delta} + \frac{1}{2}e^{-2\Phi(r_{eq})} \left[1 - \frac{b(r_{eq})}{r_{eq}} \right] V_{eff}''(r_{eq}) \delta = 0. \quad (23)$$

The condition for stability of the simple circular orbits requires $V_{eff}''(r_{eq}) > 0$ [13]. This gives

$$(3\Phi' + r\Phi'' - 2r\Phi'^2)|_{r=r_{eq}} > 0. \quad (24)$$

In terms of the tangential velocity the stability condition can be reformulated as

$$\left[\frac{d}{dr} v_{\text{tg}}^2 + \frac{2v_{\text{tg}}^2(1 - v_{\text{tg}}^2)}{r} \right] \Big|_{r=r_{eq}} > 0. \quad (25)$$

IV. TANGENTIAL VELOCITY AND REDSHIFT IN CONFORMALLY SYMMETRIC WORMHOLE GEOMETRY

In the case of the motion of a test particle in a conformally symmetric, static spherically symmetric space-time, with a non-static vector field generating the symmetry, the metric coefficient $\exp(2\Phi)$ is given by Eq. (6). Therefore for the angular velocity we find the simple expression

$$v_{\text{tg}}^2 = 1 - \frac{k}{\psi}. \quad (26)$$

Equation (26) gives a direct physical interpretation of the conformal factor ψ in terms of the tangential velocity, $\psi = k/(1 - v_{\text{tg}}^2)$. From Eq. (26) it follows that the general, physically acceptable, range of the parameter ψ is $\psi \in [k, \infty)$, corresponding to a variation of the tangential velocity between zero and the speed of light. Note that this is equivalent to stating that the expression is only valid for stable circular orbits, so that the condition $0 < k/\psi < 1$ is imposed. This condition, in particular, excludes the throat, r_0 (recall that $\psi(r_0) = 0$), which reflects the absence of stable circular orbits around the throat. Note that as the relationship (7) places a strong constraint on the specific choices of the wormhole geometries, and as the emphasis is on physically measurable quantities the conformal symmetry places severe restrictions on the range of stable orbits.

This interesting physical phenomenon can be explained on the grounds that in the wormhole geometry the instability develops due to the violation of the null energy

condition $\rho + p_r < 0$. This reflects that gravity becomes repulsive in these regimes, and the repulsive force destabilizes the orbit, where the angular momentum cannot balance with the gravitational interaction.

In terms of the conformal factor the stability condition given by Eq. (25) can be reformulated as

$$k \left[\frac{d\psi}{dr} + \frac{2}{r}(\psi - k) \right] \Big|_{r=r_{eq}} > 0. \quad (27)$$

On the other hand, the form function $b(r)$ can also be expressed as a function of the tangential velocity only:

$$b(r) = r \left[1 - \frac{k^2}{(1 - v_{\text{tg}}^2)^2} \right]. \quad (28)$$

The condition of the NEC violation can be formulated in terms of the tangential velocity of test particles as

$$\rho(r) + p_r(r) = \frac{1}{r^2} \left[\frac{2v_{\text{tg}}^2}{1 - v_{\text{tg}}^2} - r \frac{d}{dr} \frac{1}{(1 - v_{\text{tg}}^2)^2} \right] < 0. \quad (29)$$

We emphasize that Eqs. (28) and (29) are only valid for stable circular orbits in the range $0 < k/\psi < 1$. Thus, they do not apply, in particular, to the wormhole throat r_0 , where the conformal factor obeys $\psi(r_0) = 0$.

The relationship (29) is equivalent with the following condition which must be satisfied by the tangential velocity of a test particle in a wormhole geometry:

$$\frac{dv_{\text{tg}}}{dr} > \frac{1}{2r} v_{\text{tg}} (1 - v_{\text{tg}}^2)^2. \quad (30)$$

Note that this relationship is only valid if we impose that exotic matter is threaded throughout the wormhole. By integrating this equation we obtain the physical condition defining the conformally symmetric wormhole geometry as

$$v_{\text{tg}}^2 > \frac{r/R_0}{1 + r/R_0}, \quad (31)$$

where $R_0 > 0$ is an arbitrary length scale (a constant of integration).

Finally, we consider the possibility of observationally testing wormhole geometries via the observation of the frequency shifts of the light emitted by particles orbiting wormholes. Depending on the direction of motion with respect to an observer located at infinity, the radiation emitted by particles moving in circular orbits on both sides of the central region of the wormhole will be successively red shifted (in the case of the recessional motion) and blue shifted (in the case of the particle approaching the observer), respectively. Consider two observers O_E and O_∞ , with four-velocities u_E^μ and u_∞^μ , respectively. Observer O_E corresponds to the light emitter (i.e., to the particles in stable circular orbits orbiting around a wormhole and located at a point P_E of the space-time), and O_∞ represents the detector at point P_∞ , located far

from the emitter, and that can be idealized to correspond to “spatial infinity”.

The light signal travels to the observer on null geodesics with tangent vector k^μ . We may restrict k^μ to lie on the equatorial plane $\theta = \pi/2$ of the wormhole, without a significant loss of generality, and evaluate the frequency shift for a light signal emitted from O_E in a circular orbit and detected by O_∞ . The frequency shift associated to the emission and detection of the light signal is given by

$$z = 1 - \frac{\omega_E}{\omega_\infty}, \quad (32)$$

where $\omega_I = -k_\mu u_I^\mu$, and the index I refers to emission ($I = E$) or detection ($I = \infty$) at the corresponding space-time point [13]. Two frequency shifts, corresponding to maximum and minimum values, are associated with light propagation in the same and opposite direction of motion of the emitter, respectively. Such shifts are frequency shifts of a receding or approaching particle, respectively. Using the constancy along the geodesic of the product of the Killing field $\partial/\partial t$ with a geodesic tangent gives the expressions of the two shifts as [13]

$$z_\pm = 1 - e^{\Phi_\infty - \Phi(r)} \frac{1 \mp \sqrt{r\Phi'}}{\sqrt{1 - r\Phi'}} = 1 - e^{\Phi_\infty - \Phi(r)} \frac{1 \mp v_{tg}}{\sqrt{1 - v_{tg}^2}}, \quad (33)$$

respectively, where $\exp[2\Phi(r)]$ represents the value of the metric potential at the radius of emission r and $\exp[2\Phi_\infty]$ represents the corresponding value of $\exp[2\Phi(r)]$ for $r \rightarrow \infty$.

In terms of the conformal factor ψ the two shifts can be written as

$$z_\pm = 1 - e^{\Phi_\infty - \Phi(r)} \sqrt{\frac{\psi}{k}} \left(1 \mp \sqrt{1 - \frac{k}{\psi}} \right). \quad (34)$$

It is convenient to define two other quantities $z_D = (z_+ - z_-)/2$ and $z_A = (z_+ + z_-)/2$, given by

$$z_D(r) = e^{\Phi_\infty - \Phi(r)} \frac{\sqrt{r\Phi'}}{\sqrt{1 - r\Phi'}} = e^{\Phi_\infty - \Phi(r)} \frac{v_{tg}}{\sqrt{1 - v_{tg}^2}} = e^{\Phi_\infty - \Phi(r)} \sqrt{\frac{\psi}{k}} - 1, \quad (35)$$

$$z_A(r) = 1 - \frac{e^{\Phi_\infty - \Phi(r)}}{\sqrt{1 - r\Phi'}} = 1 - \sqrt{\frac{\psi}{k}} e^{\Phi_\infty - \Phi(r)}, \quad (36)$$

respectively, which can be easily connected to the astronomical observations [13]. z_A and z_D satisfy the relation $(z_A - 1)^2 - z_D^2 = \exp[\Phi_D - \Phi(r)]$, and thus in principle both $\exp[2\Phi(r)]$ and $\psi(r)$ could be obtained directly from the observations. Finally, it should be noted that the tangential velocity can be expressed solely in terms of the shifts

$$v_{tg}^2 = \frac{z_D}{1 - z_A}. \quad (37)$$

V. GRAVITATIONAL BREMSSTRAHLUNG FROM CHARGED PARTICLES

Charged particles moving on a geodesic path in a gravitational field *do emit* an electromagnetic, bremsstrahlung type radiation, as expected from classical electrodynamics, where acceleration is the source of the electromagnetic radiation. This phenomenon was pointed out first in [14], and further investigated and corrected in [15], where the process was called electro-gravitic bremsstrahlung. However, an alternative name for this specific type of electromagnetic radiation, which is in a much wider use presently, is *gravitational bremsstrahlung*. The computation of the high-frequency radiation emitted by freely falling particles moving in circular geodesic orbits in a spherically symmetric gravitational field, and in the Kerr geometry, respectively, was performed in [16]. The properties of gravitational bremsstrahlung radiation were considered in detail, in different physical situations and geometrical frameworks, in [17].

In the present Section, we calculate the total power emitted by charged particles in geodesic motion in the conformally symmetric wormhole geometry. The four-momentum dp^μ radiated by a charged particle with a four-velocity $u^\mu = (\gamma/\sqrt{-g_{00}}, \gamma v^i)$, where $\gamma = 1/\sqrt{1 - v^2}$ and $v^2 = \gamma_{ij} v^i v^j$, with $\gamma_{ij} = g_{ij}$ the spatial three-dimensional metric tensor, is given by [12]

$$dp^\mu = \frac{2}{3} e^2 \frac{du^\alpha}{ds} \frac{du_\alpha}{ds} dx^\mu. \quad (38)$$

The motion of the particle in the wormhole geometry takes places along the geodesics of the space-time, given by

$$\frac{du^\alpha}{ds} + \Gamma_{\rho\sigma}^\alpha u^\rho u^\sigma = 0, \quad (39)$$

and

$$\frac{du_\alpha}{ds} = \frac{1}{2} \frac{\partial g_{\gamma\delta}}{\partial x^\alpha} u^\gamma u^\delta, \quad (40)$$

respectively [12]. Therefore, the total four-momentum radiated by a charged particle in geodesic motion around a wormhole is given by

$$\Delta p^\mu = -\frac{1}{3} e^2 \int \Gamma_{\rho\sigma}^\alpha \frac{\partial g_{\gamma\delta}}{\partial x^\alpha} u^\rho u^\sigma u^\gamma u^\delta dx^\mu. \quad (41)$$

In particular, for a static metric we obtain the total radiated energy ΔE as

$$\Delta E = -\frac{1}{3} e^2 \Gamma_{\rho\sigma}^\alpha \frac{\partial g_{\gamma\delta}}{\partial x^\alpha} u^\rho u^\sigma u^\gamma u^\delta \Delta t, \quad (42)$$

and hence the total radiated power $P = \Delta E/\Delta t$ is given by

$$P = -\frac{1}{3} e^2 \Gamma_{\rho\sigma}^\alpha \frac{\partial g_{\gamma\delta}}{\partial x^\alpha} u^\rho u^\sigma u^\gamma u^\delta. \quad (43)$$

Consider first the case of the geodesic motion of a charged particle in the polar plane, $\phi = 0$, without a significant loss of generality, of the wormhole. The only component of the three-dimensional velocity is $v^\theta = \exp(-\Phi)d\theta/dt$, and the square of the velocity is $v^2 = r^2 \exp(-2\Phi)(d\theta/dt)^2 = v_{\text{tg}}^2$. Hence, we have $v^\theta = v_{\text{tg}}/r$.

Using $u^\mu = (\gamma e^{-\Phi}, 0, \gamma v_{\text{tg}}/r, 0)$, we obtain the radiated power by a charged particle in geodesic rotation in the conformally symmetric wormhole geometry as

$$P = \frac{2e^2\gamma^4}{3} \left(1 - \frac{b}{r}\right) \left(\Phi' - \frac{v_{\text{tg}}^2}{r}\right)^2. \quad (44)$$

Once the particle radiates, its energy is not conserved, and therefore Eq. (22) cannot be used, as for stable circular orbits we have $v_{\text{tg}}^2 = r\Phi'$, and therefore $P = 0$. The tangential velocity is an arbitrary input parameter. Moreover, for large distances from the wormhole, we can neglect the term v_{tg}^2 , thus obtaining for the radiated power

$$P \approx \frac{2e^2\gamma^4}{3} \left(1 - \frac{b}{r}\right) \Phi'^2 = \frac{2e^2\gamma^4}{3r^2} (\psi - k)^2, \quad (45)$$

where we have also expressed the conformal factor $\psi(r)$ in terms of form function $b(r)$ by using the relation $\psi(r) = \sqrt{1 - b(r)/r}$.

In the case of a purely radial geodesic motion of a charged massive particle in the plane $\phi = 0$ the only non-zero component of the three velocity is $v^r = \exp(-\Phi)dr/dt$, with the square given by

$$v^2 = e^{-2\Phi} \left(1 - \frac{b(r)}{r}\right)^{-1} \left(\frac{dr}{dt}\right)^2. \quad (46)$$

Therefore we obtain

$$v^r = \sqrt{1 - \frac{b(r)}{r}} v = \psi(r)v, \quad (47)$$

and consequently $u^\mu = (\gamma e^{-\Phi}, \gamma \sqrt{1 - b(r)/r} v, 0, 0)$.

Note that Eq. (46) may also be expressed in terms of the constant of motion E , given by

$$v^2 = 1 - \frac{e^{2\Phi}}{E^2}. \quad (48)$$

Finally, we obtain for the radiated power of a charged particle radially falling into the wormhole the expression

$$P = \frac{2e^2\gamma^4}{3} \left(1 - \frac{b}{r}\right) \left[\Phi'^2 - \frac{(b - rb')^2}{4r^4 (1 - b/r)^2} v^4\right]. \quad (49)$$

In terms of the conformal factor the power emitted for a particle in radial motion is

$$P = \frac{2e^2\gamma^4}{3r^2} [(\psi - k)^2 - (r\psi'v)^2]. \quad (50)$$

In the limit of large r we can neglect the term proportional to v^2 in the expression of the power, and we obtain again Eq. (45). One of the basic physical characteristics of the gravitational bremsstrahlung radiation is that it is independent of the mass of the radiating particle. Therefore the intensity of the radiation increases very rapidly with the charge of the body, and can reach very high values for macroscopic objects. If the electromagnetic power P from particles moving either on circular paths or in free fall towards the wormhole is measured, as well as their velocities v and positions r , then the conformal factor ψ can be obtained from the observational data as

$$\psi \approx k + \sqrt{\frac{3}{2}} \frac{r}{e\gamma^2} \sqrt{P}. \quad (51)$$

VI. DISCUSSIONS AND FINAL REMARKS

In the present paper we have considered the dynamics of test particles in stable circular orbits around static and spherically symmetric wormholes in conformally symmetric spacetimes. The analysis of this problem is important, because, through the observation of the shifts in the frequency of light emitted by high velocity, ultra-relativistic particles rotating around compact astrophysical objects, it may open the possibility of the observational detection of these hypothetical general relativistic objects.

In the case of the conformally symmetric wormholes, the main result of our analysis is the direct relation between a purely geometric quantity, i.e., the conformal factor ψ , and an observable quantity, the tangential velocity v_{tg} of the test particles moving in stable circular orbits around the central wormhole. Consequently, all the components of the metric tensor describing the geometrical properties of the wormhole, namely, the redshift function $\Phi(r)$ and the form function $b(r)$, can be expressed in terms of an observable physical quantity. Moreover, other directly observable quantities, namely, the shifts in the frequency of light emitted by a particle rotating around a wormhole can also be expressed in terms of the conformal factor.

Specific physical properties and characteristics of gravitational bremsstrahlung emitted by charged particles in geodesic motion in conformally symmetric wormhole geometries were also explored, and expressions relating the radiated power and the conformal factor were found. Thus, it is interesting to note that the conformal factor ψ can be obtained from the observational data, if the electromagnetic power from charged particles moving either on circular paths or in free fall towards the wormhole is measured, as well as their velocities and positions.

In light of the points emphasized above, the observation of these shifts and the radiated power from charged particles, through gravitational bremsstrahlung, could provide a direct observational test of the wormhole geometry, and implicitly, of the conformal structure of the space-time. Note that the analysis considered through-

out this work could be generalized to other general relativistic compact objects and also by imposing a non-static conformal function $\psi(r, t)$, where a wider variety of exact solutions may be found. However, this shall be analyzed in a future work.

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